On decision procedures for some systems of modal propositional logic

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In regard to decision procedures based on Kripke-models, several methods have already been found. While our method is essentially similar to the methods of Kripke (1) and Hughes-Cresswell (2), we induced it by modifying Tableaux method of E.w.Beth and R.M.Smullyan (3) (4) limited to non-modal predicate logic by using the method of S.Saito (5) and extending it so as to be applied to modal propositional logic.

A formula α of modal propositional logic is said to be logically true on a model M if and only if $V(\alpha, w)=T$ for any element of W_M. Therefore if $V(\alpha, w_1)=F$ for some $w_1 \in W$, α is not logically true on the model. Therefore if we suppose that $V(\alpha, w_1)=F$ for some $w_1 \in W$, α is logically true if inconsistency arises when valuation is given for every possible subformula of α. This method gives a test for judging whether a formula is logically true or not. This makes it possible to say that if a truth test of any formula of modal propositional logic (herinafter referred to as a formula) can be settled with a certain limited number of steps, then it means that a decision procedure has been given. In this paper we will compose a procedure of limited truth tests.

1

In this section we shall compose a truth test (hereinafter referred to as a test) for T-system of any formula, and prove the procedure to be valid. Our test is given by the following Rule 1−5:

[Rule 1] Let us suppose that the valuation-function of α (a given formula on a model) is $V(\alpha, w_1)=F$.

If in any compound formula α forms of 'α' are '$\Box\beta$', '$\sim\Box\beta$', and '$\Box\beta\vee\gamma$', then '$\Box$', '$\sim$', and '$\vee$' are called main symbols of α.

[Rule 2] Valuation-functions which have modal symbols as main symblos of the formula α are any of the following ones, and follow the under-mentioned rules. They are applied, however, so long as w in valuation-functions is a constant.

2−1 $\dfrac{V(\Box\alpha, w_i)=T}{V(\alpha, {}_iw_x)=T}$ (w_i is a constant)
(${}_iw_x$ is a variable related with w_i)

2−2 $\dfrac{V(\Box\alpha, w_i)=F}{V(\alpha, {}_iw_1)=F}$ (${}_iw_1$ is a constant related with w_i which has not been used yet.)

2−3 $\dfrac{V(\Diamond\alpha, w_i)=T}{V(\alpha, {}_iw_1)=T}$ (${}_iw_1$ is a constant related with w_i which has not been used yet.)

2−4 $\dfrac{V(\Diamond\alpha, w_i)=F}{V(\alpha, {}_iw_x)=F}$ (${}_iw_x$ is a variable related with w_i.)

Here we will explain the notation used in the above-mentioned rules. $V(\alpha, {}_iw_x)=T(F)$ is an abbreviation

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of V $(\alpha, w_x)=T(F)$ for every $w_x \in W$ which is w_iRw_x. ${}_iw_x$ is substitutable with every term related with w_i ($w_i Rw_x$). In the cases of w_iRw_j (${}_iw_j$ in abridged notation) and w_iRw_k (${}_iw_k$), for example, ${}_iw_x$ is substitutable with ${}_iw_j$ and ${}_iw_k$ respectively. This ${}_iw_x$ is called a variable related with w_i.

Accordingly, $V(\alpha, {}_iw_1)=T(F)$ is $V(\alpha, w_1)=T(F)$. ${}_iw_1$ is an abbreviation of w_iRw_1 (for some $w_1 \in W$) . If ${}_iw_1$ has already been used in another place, we use ${}_iw_2$ in stead of ${}_iw_1$. Similarly we use ${}_iw_3$ if ${}_iw_2$ has already been used. ${}_iw_1$ (for some $w_1 \in W$) is a specific term of an unknown W, related with w_i, and is a constant related with w_i which has not been used yet. It is evident that Rule 2 is proued to be valid by the definition of valuation-function.

[Rule 3] Valustion-functions which have truth-function ($\sim$, $\vee$, $\wedge$, $\supset$, $\equiv$) as main symbols of a formula are any of the following ines, and the under-mentioned rules are applied to them. However they are applied so long as w in valuation-function is a constant.

3－1 $\dfrac{V(\sim\alpha, w_i)=T}{V(\alpha, w_i)=F}$ $\qquad$ $\dfrac{V(\sim\alpha, w_i)=F}{V(\alpha, w_i)=T}$

3－2 $V(\alpha\vee\beta, w_i)=T$
$V(\alpha, w_i)=T, \quad V(\beta, w_i)=T$

$\dfrac{V(\alpha\vee\beta, w_i)=F}{V(\alpha, w_i)=F}$
$V(\beta, w_i)=F$

3－3 $\dfrac{V(\alpha\wedge\beta, w_i)=T}{V(\alpha, w_i)=T}$
$V(\beta, w_i)=T$

$V(\alpha\wedge\beta, w_i)=F$
$V(\alpha, w_i)=F, \quad V(\beta, w_i)=F$

3－4 $V(\alpha\supset\beta, w_i)=T$
$V(\alpha, w_i)=F, \quad V(\beta, w_i)=T$

$\dfrac{V(\alpha\supset\beta, w_i)=F}{V(\alpha, w_i)=T}$
$V(\beta, w_i)=F$

3－5 $V(\alpha\equiv\beta, w_i)=T$
$V(\alpha, w_i)T, \quad V(\alpha, w_i)=F$
$V(\beta, w_i)=T, \quad V(\beta, w_i)=F$

$V(\alpha\equiv\beta, w_i)=F$
$V(\alpha, w_i)=T, \quad V(\alpha, w_i)=F$
$V(\beta, w_i)=F, \quad V(\beta, w_i)=T$

When Rules 2 and 3 cannot be applied, α (a formula in a valuation-function) includes ${}_iw_x$ (a variable related with w_i) so long as it is not an atomic formula.

[Rule 4] Only if Rules 2 and 3 cannot be applied, ${}_iw_x$ is substituted with every constant related with w_i which appears during the process of Tableaux in a variable w_x in valuation-function, contants related with w_i which are obtained according to the conditions of R in a model being included. However, if there is a contradiction in the same branch, it is needless to substitute it with another constant.

The condition of R on T-model is only a reflexive relation, so ${}_iw_i$ corresponding to ${}_iw_x$ is obtained. This testing method is used not only for T-system but also B-system and D-system.

[Rule 5] Let us resolve any formula α for the model into valuation-functions of atomic formulae by applying Rules 1 to 4 a limited number of times. A given formula α is decided to be logically true on the model if it is inconsistent in every resolved branch of Tableaux.

And it is decided not to be logically true on the model if it is not inconsistent at least in one branch.

Now we demonstrate the method of applying the above-mentioned rules.

Example 1 $\Box p\supset p$ is logically true on T-model.

(1)	$V(\Box p\supset p, w_1)=F$	Rule 1
(2)	$V(\Box p, w_1)=T$	(1), Rule 3－4
(3)	$V(p, w_1)=F$	(1), Rule 3－4
(4)	$V(p, {}_1w_x)=T$	(2), Rule 2－1
(5)	$V(p, w_1)=T$	(4), Rule 4, Condition R

× ((3), (5) Inconsistent)

$\therefore \Box p \supset p$ is logically true on T-model. (Rule 4)

The formula (5) is gained by substituting ${}_1w_x$ of formula (4) with ${}_1w_1$ because ${}_1w_x$ is ${}_1w_1$ according to the condition of R on T-model i.e. the reflexive relation $(w_x R w_x)$.

Example 2 $\Box(p \vee \Diamond q) \supset (\Box p \vee \Diamond q)$ is not logically true on T-model.

(1) $V(\Box(p \vee \Diamond q) \supset (\Box p \vee \Diamond q), w_1) = F$ Rule 1
(2) $V(\Box(p \vee \Diamond q), w_1) = T$ (1), Rule 3—4
(3) $V(\Box p \vee \Diamond q, w_1) = F$ (1), Rule 3—4
(4) $V(p \vee \Diamond q, {}_1w_x) = T$ (2), Rule 2—1
(5) $V(\Box p, w_1) = F$ (3), Rule 3—2
(6) $V(\Diamond q, w_1) = F$ (3), Rule 3—2
(7) $V(p, {}_1w_2) = F$ (5), Rule 2—2
(8) $V(q, {}_1w_x) = F$ (6), Rule 2—4
(9) $V(p \vee \Diamond q, w_1) = T$ (4), Rule 3, Condition R
(10) $V(p, w_1) = T$, (11) $V(\Diamond q, w_1) = T$ (9), Rule 3—2
(14) $V(p \vee \Diamond q, {}_1w_2) = T$ (4), Rule 4 (12) $V(q, {}_1w_3) = T$ (11), Rule 2—3
(13) $V(q, {}_1w_3) = F$ (8), Rule 4, Condition R
× ((12), (13) Inconsistent)
(15) $V(p, {}_1w_2) = T$, (16) $V(\Diamond q, {}_1w_2) = T$ (14), Rule 3—2
× ((7), (15) Inconsistent) (17) $V(q, {}_2w_4) = T$ (16), Rule 2—3
(Not inconsistent)

$\therefore \Box(p \vee \Diamond q) \supset (\Box p \vee \Diamond q)$ is not logically true on T-model. (Rule 5)

The formula (9) is obtained as a result of substituting of the formula (4) with ${}_1w_1$ (the reflexive relation of R). Among the first ten formulae there are two formulae in which valuation-function of atomic formula p appears. They, the formulae (7) and (10), are not inconsistent. Because the two w-s are not the same: the one is w_2 in the formula (7) and the other is w_1 in the formula (10), and they are not inconsistent even though their truth-values of p (i.e. F and T) are different from each other. Thereupon the test can be continued. As there exists a new constant w_2 related with w_1 (i.e. ${}_1w_2$) ine the formula (7), the formula (14) is obtained by substituting ${}_1w_x$ in the formula (4) with ${}_1w_2$. (Rule 4 makes it necessary to substitute ${}_1w_x$ with every term related with w_1.) If you go up the branch of the formula (17), you will find a valuation-function corresponding to the atomic formula q in the formula (8). Though w_4 newly appears in the formula (17), w_x in the formula (8) cannot be substituted with it. Because w_4 is a constant related with w_2, not a constant related with w_1. This indicates that it is inconsistent on S4-model in possession of the condition of R in which there exists ${}_1w_2$, ${}_2w_4$, ${}_1w_4$ (transitive relation) and, accordingly, that the given formula is logically true on S4-model.

The following theorem can be formed, which shows that these rules are valid ones.

[THEOREM 1] The procedure in Rules 1 to 5 is decision procedure in system-T, system-B and system-D.

[PROOF] It is sufficient if the following two theorems can be demonstrated.

1 When F is assigned to valuation-functions V (α, w_1) for some $w_1 \in W$ in the given formula α, it is logically true if inconsistency arises in all the branches concerning the assignment of truth-value, which is constructed by a limited procedure.

2 If inconsistency does not arise at least in one branch, then the given formula α is not logically true, which is also contructed by a limited procedure.

[PROOF OF 1]

That the valvation of the formula for some $w_i \in W$ put to the test is "F" makes it necessary that valuation can be formed in any branch of the test. (Therefore, if inconsistency arises in every branch of the test, then the truth-value of the formula put to the test connot be "F".

Now we will prove that valuations are given for atomic formulae by limited times of application of Rules 1 to 4. It is clear that the number of the branches which arisis where Rule 3 is applied is limited because the length of the formula is limited. Therefore, all that we have to do is to show that the number of the formulae is limited which are obtained by substituting ${}_iw_x$ produced by Rules 2−1 and 2−4 with every constant related with w_i produced through application of Rule 4.

As the length of the formula is limited, it is only when Rules 2−2 and 2−3 are applied that new constants related with w_i arise. Thus the number of the new constants related with w_i is limited. (In T-, B-, and D -models the conditions of R includes nothing other than reflexive relation and symmetric relation.) Therefore, it is also limited, the numbers of the valuation-functions obtained by substituting ${}_iw_x$.

[PROOF OF 2]

It requires us to show that the valuation of the formula put to the test is "F" if we give a certain suitable valuation to the atomic formula included in the formula put to the test.

Based upon a branch not inconsistent with the value-assignment for an atomic formula, let us give valuations to all the atomic formulae included in the formula put to the test as follows.

Let valuations hold for all the atomic formulae in the branch. Concerning any formula α arising in the branch, let the constants related with w_a of valuation-functions $v(\alpha, {}_aw_i)$, $V(\alpha, {}_aw_j)$, $\cdots V(\alpha, {}_aw_l)$ arising in the branch be ${}_aw_i$, ${}_aw_j$, $\cdots {}_aw_l$. If ${}_aw_x$ differs from every one of ${}_aw_i$, ${}_aw_j$, $\cdots {}_aw_l$, then let the valuation for ${}_aw_x$ of α be the same as the valuation of ${}_aw_i$. And if ${}_aw_x$ is the same as any of ${}_aw_i$, ${}_aw_j$, $\cdots {}_aw_l$, then let the valuation for ${}_aw_x$ of α be the same as valuation of any of them. Now we will show that if we go on conducting value-assingnment conversely on the basis of a branch whose value-assignment for atomic formulae is not inconsistent, then we can also trace back Rules 2−1 and 2−4.

To trace back value-assignmemt is unique where Rules 2 and 3 are used. We treat Rule 4 as follows.

Let it be supposed that Rule 4 is used upon the valuation for ${}_aw_x$ of any formula, $\alpha(=f(p, q, \cdots))$, including atomic formulae p, q,⋯.

If $V(f(p, q, \cdots), {}_aw_x) = T(F)$, then the following valuation-functions appear in the steps of $V(f(p, q, \cdots), {}_aw_x) = T(F)$ and the succeeding ones.

$V(f(p, q, \cdots), {}_aw_i) = T(F)$

$V(f(p, q, \cdots), {}_aw_j) = T(F)$

• • •

$V(f(p, q, \cdots), {}_aw_l) = T(F)$

${}_aw_i$, ${}_aw_j$, $\cdots$ ${}_aw_l$ are the constants related with w_a included in the valuation-functions for the atomic formulae p, q, ⋯ as the arguments of f. If ${}_aw_x$ in $V(f(p, q, \cdots), {}_aw_x)$ is one of ${}_aw_i$, ${}_aw_j$, $\cdots$ ${}_aw_l$, then all of $V(f(p, q, \cdots), {}_aw_i)$, $V(f(p, q, \cdots), {}_aw_j)$, $\cdots V(p, q, \cdots), {}_aw_l)$ have the same valuation, that of $V(f(p, q), {}_aw_x)$. And when ${}_aw_x$ differs from ${}_aw_i$, ${}_aw_j$, ${}_aw_l$, the valuations of $V(p, {}_aw_x)$ and $V(q, {}_aw_x)$ are $V(f(p, q, \cdots), {}_aw_x) = V(f(p, q, \cdots), {}_aw_i)$ anyway, for they are in accordance with the valuations of $V(p, {}_aw_i)$, $V(q, {}_aw_j)$, $\cdots$ according to the previous agreement. Therefore we can also trace back conversely concerning Rule 4. In each of these steps "F" is assigned to logically true formulae tested a limited times as well as in Rule 1.

2

Let us study the method of truth tests on the models S4 and S5.

Let us apply the method mentioned in 1 to the following example.

Example 3 $\Diamond\Box$ p is not logically true on S4-model.

(1)	$V(\Diamond\Box p, w_1)=F$	Rule 1
(2)	$V(\Box p, {}_1w_x)=F$	(1), Rule 2−4
(3)	$V(\Box p, {}_1w_1)=F$	(2), Rule 4, Condition R
(4)	$V(p, {}_1w_2)=F$	(3), Rule 2−2
(5)	$V(\Box p, {}_1w_2)=F$	(2), (4), Rule 4
(6)	$V(p, {}_2w_3)=F$	(5), Rule 2−2
(7)	$V(\Box p, {}_1w_3)=F$	(2), (6), Rule 4, Condition R
(8)	$V(p, {}_3w_4)=F$	(7), Rule 2−2
(9)	$V(\Box p, {}_1w_4)=F$	(2), (8), Rule 4
• • •		
(n)	$V(p, {}_{n-5}w_{n-4})=F$	
(n+1)	$V(\Box p, {}_1w_{n-4})=F$	
• • •		

The formula 3 is gained by substituting ${}_1w_x$ in the formula (2) with w_1 according to reflexive relation. The formulae (5) and (7) are respectively gained by substituting ${}_1w_x$ in the formula (2) with ${}_1w_2$ in the formula (4) , and ${}_1w_x$ with ${}_2w_3$ in the formula (6) according to transitive relation (${}_iw_j, {}_jw_k \Rightarrow {}_iw_k$). In the same way new relative terms arises successively.

On models, such as Example 3, stronger than S4-model there can be cases in which test is endless according to transitive relation of Condition R. Therefore Rules 1 to 5 are not decision procedure of S4.

However we can gain the decision procedure of S4 by modifying Rule 4. We prepare the following terms in order to make our explanation clear.

In the formula of valuation-function $V(\alpha, w)=e$ ($e=T$ or F), α and w are respectively called argument formula of the formula of valuation-function and argument of the formula of valuation-function. And e is simply called value of the formula of valuation-function.

Now let us explain Example 3.

The formulas (1) and (3), (4) and (5), and (6) and (7) respectively have the same argument w_1, w_2, and w_3. Thus the formulas of valuation-function (n+3) and (n+4) have the same w_n. All the argument formulae in the formulae of valuation-function each of which has argument $w_2, w_3, \cdots, w_n, \cdots$are p and $\Box$ p, the same subformulae of the given formula $\Diamond\Box$ p. No formulae do not exist which are not its subformulae.

However w_1 is different from $w_2, w_3, \cdots, w_n$: The argument formula of the formula of valuation-function possessed of w_1 include "$\Diamond\Box$ p", but not "p". And the values of the argument formula p for $w_2, w_3, \cdots$are all "F", and the values of $\Box$ p for $w_2, w_3, \cdots, w_n$ are all "F".

Thus in the group of the formulae of valuation-function whose argument is w_2, such as (4) and (5), and the group of those whose argument is w_3, such as (6) and (7), the values of p, $\Box$ p are respectively the same although the argument of the former w_2 is different from that of the latter w_3. In such a case as this it is unnecessary to substitute ${}_1w_x$ in the formula (2) with the derived terms ${}_1w_2, {}_1w_3, {}_1w_4, \cdots$. It will be proved in the following theorem. Therefore it is possible to construct the test on S4-model with limited number of steps by modifying Rule 4, and the following decision procedure of S4 is gained.

Our new testing method is made up through adding the following restrictive conditions to Rule 4 (in which all the relative terms are substituted).

[Rule 4′]

In S4-model and the stronger ones the following restrictive conditions are added to Rule 4.

Restrictive conditions: If the formula of valuation-function possessed of the argument ${}_{i}w_{x}$, $V(\beta, {}_{i}w_{x})=e$, appears in Tableaux of the test, ${}_{i}w_{x}$ in the original formula $V(\beta, {}_{i}w_{x})=e$ must not be substituted with the new relative term ${}_{i}w_{j}$ derived from w_i in the following case.

As for the formulae in which w_i, ⋯, w_j can be substituted among the subformulae (α_1, α_2, ⋯, α_n, α_1', α_2', ⋯) of the given formula α being tested, if.

$V(\alpha_1, w_i)=e_1$
$V(\alpha_2, w_i)=e_2$
⋮
$V(\alpha_n, w_i)=e_n$

$V(\alpha_1', w_j)=e_1$
$V(\alpha_2', w_j)=e_2$
⋮
$V(\alpha_n', w_j)=e_m$,

then $n=m$, $\alpha_1=\alpha_1'$, ⋯, $\alpha_n=\alpha_n'$, and $e_1=e_1'$, ⋯, $e_n=e_n'$.

Example 4 $\Box\Diamond p\supset\Diamond\Box p$ is not logically true on S4-model.

(1)	$V(\Box\Diamond p\supset\Diamond\Box p, w_1)=F$	Rule 1
(2)	$V(\Box\Diamond p, w_1)=T$	(1), Rule 3−4
(3)	$V(\Diamond\Box p, w_1)=F$	
(4)	$V(\Diamond p, {}_{1}w_{x})=T$	(2), Rule 2−1
(5)	$V(\Box p, {}_{1}w_{x})=F$	(3), Rule 2−4
(6)	$V(\Diamond p, w_1)=T$	(4), Rule 4′
(7)	$V(\Box p, w_1)=F$	(5), Rule 4′
(8)	$V(p, {}_{1}w_{2})=T$	(6), Rule 2−3
(9)	$V(\Diamond p, {}_{1}w_{2})=T$	(4), (8), Rule 4′
(10)	$V(\Box p, {}_{1}w_{2})=F$	(5), (8), Rule 4′
(11)	$V(p, {}_{2}w_{3})=T$	(9), Rule 4′
(12)	$V(\Diamond p, {}_{1}w_{3})=T$	(4), Rule 4′
(13)	$V(\Box p, {}_{1}w_{3})=F$	(5), Rule 4′
(14)	$V(p, {}_{2}w_{4})=F$	(10), Rule 4′
(15)	$V(\Diamond p, {}_{1}w_{4})=T$	(4), Rule 4′
(16)	$V(\Box p, {}_{1}w_{4})=F$	(5), Rule 4′
(17)	$V(p, {}_{4}w_{5})=T$	(15), Rule 4′
(18)	$V(\Diamond p, {}_{1}w_{5})=T$	(4), Rule 4′
(19)	$V(\Box p, {}_{1}w_{5})=F$	(5), Rule 4′
(20)	$V(p, {}_{4}w_{6})=F$	(16), Rule 4′
(21)	$V(\Diamond p, {}_{1}w_{6})=T$	(4), Rule 4′
(22)	$V(\Box p, {}_{1}w_{6})=F$	(5), Rule 4′

⋮

The formula (6) is gained by applying the reflexive relation of R to ${}_{1}w_{x}$ in the formula (4) according to Rule 4′. Based on the formula (6), the formulas (8) and (10) are gained by substituting ${}_{1}w_{x}$ in the formulae (4) and (5) with ${}_{1}w_{2}$ of the formula (8). Therefore (only p, ◇p, and □p are the) argument formulae possessed of w_2 as argument among the subformulas of the given formula.

The formulae of valuation-function possessed of w_3 as argument are the formulae (11), (12) and (13), and (8) and (11), (9) and (12), and (10) and (13) have the same value respectively according to Rule 4′. Therefore it is unnecessary to substitute ${}_1w_x$ in formulae (4) and (5) with the relative terms derived after w_3.

As for the formulae of valuation-function possessed of w_4 as argument, however, there are ones whose values are different whereas whose argument formulae are the same, e.g. (8) and (14). Therefore we take another step and gain ${}_4w_5$ from the formula (15) and ${}_4w_6$ from the formula (16), which are the relative terms derived from w_4. Now (17), (18) and (19), the formulae of valuation-function possessed of w_5 as argument, respectively have the same values as those of (8), (9) and (10), the formulae of valuation-function possessed of w_2 as argument, so it is unnecessary to substitute ${}_1w_x$ in the formulae (4) and (5) with the derived (relative) terms after w_5 according to Rule 4′. Owing to the same reason (20), (21) and (22), the formulae of valuation-function possessed of ${}_4w_6$ as argument, also have the same values respectively as those of (14), (15) and (16), the formulae of valuation-function possessed of w_4 as argument. Here ends the test concerning to all the relative terms derived from (6).

It shows that all we have to do is put a certain number of arguments to the test in case of the relative terms derived from the formula (7), too. The relative terms derived from w_1 as follws.

$$\begin{tikzcd}
 & & w_3 & \\
w_1 \arrow[r] & w_2 \arrow[ur] \arrow[dr] & & w_5 \\
 & & w_4 \arrow[ur] \arrow[dr] & \\
 & & & w_6
\end{tikzcd}$$

As the formulae of valuation-function respectively possessed of w_2 and w_3 as argument have the same value ($w_2 \Leftrightarrow w_3$ is the indication of that), and as $w_2 \Leftrightarrow w_5$ and $w_4 \Leftrightarrow w_6$, so it is unnecessary to put to the test the derived terms after w_3, w_5, and w_6. When we put to the test the terms derived from the formula (7) in the same way, we gain the followings.

$$\begin{tikzcd}
 & & w_3' & \\
w_1 \arrow[r] & w_2' \arrow[ur] \arrow[dr] & & w_5' \\
 & & w_4' \arrow[ur] \arrow[dr] & \\
 & & & w_6'
\end{tikzcd}$$

$w_{2'} \Longleftrightarrow w_{3'}$, $w_{2'} \Longleftrightarrow w_{5'}$, $w_{4'} \Longleftrightarrow w_{6'}$

The result is as follows.

$$\begin{tikzcd}
 & w_2 \arrow[r] & w_4 \\
w_1 \arrow[ur] \arrow[dr] & & \\
 & w_2' \arrow[r] & w_4'
\end{tikzcd}$$

We can summarize the above-mentioned results of the tests as follows.

(1)	V(□◇p⊃◇□p, w_1)=F	
(2)	V(□◇p, w_1)=T	} (1)
(3)	V(◇□p, w_1)=F	
(4)	V(◇p, ${}_1w_x$)=T	(2)
(5)	V(□p, ${}_1w_x$)=F	(3)
(6)	V(◇p, w_1)=T	(4)
(7)	V(□p, w_1)=F	(5)
(8)	V(p, ${}_1w_2$)=T	(6)
※(9)	V(◇p, ${}_1w_2$)=T	(4)
(10)	V(□p, ${}_1w_2$)=F	(5)
(11)	V(p, ${}_2w_4$)=F	(10)
※(12)	V(◇p, ${}_1w_4$)=T	(4)
※(13)	V(□p, ${}_1w_4$)=F	(5)

(14) $V(p, {}_1w_2)=F$ (7)

(15) $V(\Diamond p, {}_1w_2)=T$ (4)

※(16) $V(\Box p, {}_1w_2)=F$ (5)

(17) $V(p, {}_2w_4)=T$ (15)

※(18) $V(\Diamond p, {}_1w_4)=T$ (4)

※(19) $V(\Box p, {}_1w_4)=F$ (5)

※ These formulae produce only the formulae of valuation-function of the same value if developed further. So the following calculation is not necessary for them according to the restrictive condition of Rule 4.

Example 5 $\Diamond(p\vee\Box\sim p)$ is logically true on S4-model.

(1) $V(\Diamond(p\vee\Box\sim p), w_1)=F$

(2) $V(p\vee\Box\sim p, {}_1w_x)=F$ (1)

(3) $V(p\vee\Box\sim p, {}_1w_1)=F$ (2)

(4) $V(p, w_1)=F$ (3)

(5) $V(\Box\sim p, w_1)=F$ (3)

(6) $V(\sim p, {}_1w_2)=F$ (5)

(7) $V(p, {}_1w_2)=T$ (6)

(8) $V(p\vee\Box\sim p, {}_1w_2)=F$ (2), (6)

(9) $V(p, {}_1w_2)=F$ (8)

(10) $V(\Box\sim p, {}_1w_2)=F$ (8)

(7) and (9) are contradictory to each other. Therefore the given formula is logically true.

[THEOREM 2] Procedure in Rules 1, 2, 3, 4′, and 5 is decision procedure on S4(S5)-model.

[PROOF] Difference between Theorem 1 and Theorem 2 is that between Rule 4 and Rule 4′. Therefore this theorem can be proved through indicating that Rule 4′ is a valid one. Furthermore it can be indicated through showing that the restrictive condition of Rule 4′ is valid in the test on S4-model.

First we show that the test is not endless.

Even though the number of derived relative terms is unlimited, that of argument formulae possessed of the same w_l as argument is limited. This is because every argument formula is one of the subformulae in the given formula (the first formula being tested). And the combination of the values of formulae of valuation-function possessed of the same w_l as argument is limited in number. (If the number of the subformulae in the given formula is n, then the combination of the formulae of valuation-function are no more than 2^n in all.) Therefore the value of the formula of valuation-function possessed of w as argument excluding the limited number is the same as that of the formulae of valuation-function possessed of either argument.

Secondly let us examine a case in which formulae of valuation-function possessed of ${}_aw_x$ as argument appear in the test.

Let the two relative terms derived from w_a be ${}_aw_l$ and ${}_aw_j$. And if we gain all the formulae of valuation function respectively possessed of w_l and w_j as argument as follows,

$$\mathrm{I}\begin{cases} V(\beta_1, w_l)=e_1 \\ V(\beta_2, w_l)=e_2 \\ \cdot\quad\cdot\quad\cdot \\ V(\beta_n, w_l)=e \end{cases} \qquad \mathrm{II}\begin{cases} V(\beta_1, w_j)=e_1 \\ V(\beta_2, w_j)=e_2 \\ \cdot\quad\cdot\quad\cdot \\ V(\beta_n, w_j)=e_n \end{cases}$$

then it is unnecessary to substitute ${}_aw_x$ with derived relative terms after w_j according to the restrictive condition. We will show that such a test is unnecessary.

Let an arbitrary relative term further derived from w_j be ${}_j w_k$.

$$\text{III}\begin{cases} V(\gamma_1, w_k) = e_1' \\ V(\gamma_2, w_k) = e_2' \\ \quad \cdot \quad \cdot \quad \cdot \\ V(\gamma_m, w_k) = e_m' \end{cases}$$

The formulae in the group III are gained by developing those in the group II. Therefore if $V(\beta_e, w_j) = e_e$ is a formula gained through developing, then it is $V(\beta_e, w_l) e_e$ according to the supposition. Thus the formula of valuation-function gained through substituting ${}_a w_x$ with w_k is the same as that gained through developing $V(\beta_e, w_l) = e_e$.

Let the derived term of $V(\beta_e, w_l) = e_e$ be w_e, and the following formulae are gained.

$$\text{IV}\begin{cases} V(\gamma_1, w_l) = e_1' \\ V(\gamma_2, w_l) = e_2' \\ \quad \cdot \quad \cdot \quad \cdot \\ V(\gamma_m, w_l) = e_m' \end{cases}$$

Now we have shown that if all the formulae of valuation-function respectively possessed of w_i and w_j as argument have the same value, then it is unnecessary to put to the test the terms derived after them. The above-mentioned shows that the restrictive condition of Rule 4′ is valid, and proves Theorem 2 to be true.

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